Journal of Structural Geology, Vol. 12, No. 1, pp. 131 to 137, 1990
Printed in Great Britain

0191–8141/90 \$03.00+0.00

Determination of the strain ellipsoid from sectional data

DECLAN G. DE PAOR

Department of Earth & Planetary Sciences, Johns Hopkins University, Baltimore, MD 21218, U.S.A.

(*Received* 21 *March* 1989; *accepted in revised form* 7 *June* 1989)

Abstract—This paper presents a new way to determine the shape and orientation of the triaxial strain ellipsoid, given sectional strain data from at least three arbitrary planes. First the sectional strain ellipses are scaled to maximize compatibility along their lines of intersection. Then, using formulae for strain determination from three known stretches, the sectional strain ellipse is calculated for a 'fourth' plane chosen to intersect the data planes at the highest possible angles. By repeated sectional strain determination in a set of test planes oriented perpendicular to the above fourth plane, the triaxial strain state is revealed. The longest test sectional strain ellipse long axis is the strain ellipsoid's long axis and the shortest test sectional strain ellipse short axis is the strain ellipsoid's short axis. The strain ellipsoid's intermediate axis is the pole to the plane of maximal and minimal stretches and its stretch is calculated in the test plane in which it lies.

Manual calculations, while easily understood, are time-consuming, requiring several hours for one triaxial strain determination, but a set of computer programs is available from the author. Accuracy is evaluated by determining the strain state in the data planes given the calculated triaxial ellipsoid's principal sections, and comparing results of such determinations with the observed data.

INTRODUCTION

MOST strain analysis techniques yield only two-dimensional data, so to derive three-dimensional strain it is necessary to combine ellipses from three or more measurement planes. Despite considerable literature on the subject (Ramsay 1967, Helm & Siddans 1971, Ramberg 1976, Shimamoto & Ikeda 1976, Oertel 1978, Milton 1980, Gendzwill & Stauffer 1981, Owens 1984, De Paor 1986, 1988, Treagus 1986), combining two-dimensional data to give a three-dimensional solution remains a major impediment in the way of more widespread practical applications of strain theory. Most procedures described in the literature involve a level of computation that precludes frequent use by field geologists.

Furthermore, whereas any section of an ellipsoid is an ellipse, the reverse is not true: a set of arbitrary ellipses do not necessarily lie on the surface of any real ellipsoid—they could lie on a cylinder, paraboloid or hyperboloid, for example, or they might not be confined within a simply connected surface in the topological sense. Since errors are inevitable in determination of sectional strain data, it has always been necessary to render natural data compatible by adjusting sectional ellipses to achieve a perfect fit, either by modifying calculated axes directly or by adjusting components of the reciprocal quadratic tensor whose eigenvectors determine those axes. Field geologists are rightly sceptical of such mathematical 'fudge factors'. Even after modification of the data, complex numbers arise in calculations when the ellipses are really sections of a quadric other than an ellipsoid.

The purpose of this paper is to present a simple solution to the problem using a readily-understood method. The same steps are followed in manual and computerized computations. Thus, the computer speeds the process without obscuring the logic employed. Even if restricted to the use of a hand calculator, the task can easily be completed in an afternoon.

OUTLINE OF THE METHOD

Given the minimum of three data planes, the method involves the following steps: (i) determination of orientations, shapes and relative sizes of three aribtrary sections of the strain ellipsoid; (ii) careful choice of a fourth plane intersecting the three data planes in a tetrahedron with large dihedral angles; (iii) calculation of the strain state in the fourth plane using formulae (De Paor 1988) for strain determination from three known stretches; (iv) determination of the strain state in a large set of test planes perpendicular to the fourth plane; (v) inspection of these test strain ellipses to reveal absolute maximum and minimum radii which are identified as the strain ellipsoid's longest and shortest semiaxes; the intermediate ellipsoid axis is located perpendicular to the plane of the maximum and minimum and its length is determined using the test plane in which it lies.

That concludes the theory of the method; however, a number of practical problems arise in the implementation of this simple plan.

Determination of sectional strain ratios and orientations

Sectional strain data may be collected from joint surfaces in the field or from oriented sections. It may not be convenient, and it is not necessary, to obtain mutually perpendicular sections, but to avoid inaccuracies, dihedral angles should be large—ideally in the range 90° ± 20°. Reliable results cannot be expected if section planes are almost parallel! The laboratory procedure begins by placing an oriented specimen in a bowl of sand and restoring its field orientation. The section dips are recorded in the range [0°,90°] while strikes in the range [0°,360°] are designated by viewing along the strike line in the direction that makes the dip a clockwise deflection from the horizontal. Strain is determined mesoscopically or microscopically using any selected technique. Strain ratios and axial pitches are calculated—the latter as positive clockwise angles in the range [0°,180°] relative to the quoted end of the section's strike line.

Determination of the sizes of sectional ellipses

Whilst some methods of strain analysis give absolute values of the sectional ellipse axes S_1 and S_2 or, equivalently, their reciprocal quadratic stretches λ_1' and λ_2', most only permit calculation of an axial ratio R_s. Principal stretches are given by the formulae

$$S_1 = \sqrt{R_s A_s} \qquad S_2 = \sqrt{\frac{A_s}{R_s}}, \tag{1}$$

where A_s is termed area stretch. Since A_s is not detected during most strain analysis, the stretches of arbitrary lines in data planes are determined only to within a common multiplication factor. Solution of the standard ellipse equation

$$\lambda' = \lambda_1' \cos^2 \phi + \lambda_2' \sin^2 \phi \tag{2}$$

is possible only if $A_s = 1$ is assumed. As a consequence, the two stretch estimates available for each of the specimen edges (lines of intersection AB, BC and CA of the data planes) are certain to differ. One ellipse may lie within another like the rings of a universal stage. Ellipse B may be scaled up or down to intersect ellipse A by multiplying both of its axes by a common area stretch factor A_b determined by the ratio of the two incompatible stretches along the edge AB, and similarly ellipse C may be made to intersect ellipse B, but then ellipses A and C will generally not intersect; rather one will pass within the orbit of the other. This 'closure error' may result from inaccurate strain measurement or from a breakdown in the assumptions of material continuity and strain homogeneity. Left uncorrected, it causes previously published methods of triaxial strain determination to yield multiple strain ellipsoid estimates or even to output complex numbers. In the method presented here, if A_s is unknown, the user minimizes closure errors by scaling. Of course, if the incompatibilities are very large, then the method will not give accurate results, and therefore careful attention must be paid to minimization and equitable distribution of closure errors. A mathematical algorithm has been described by Milton (1980) and a graphical procedure by De Paor (1986). However it is more useful to employ the computer program "2D–3D Compatibility" described in the Appendix in order to increase speed and accuracy.

The fourth plane

Having determined three sectional ellipse sizes to within a common scaling factor by minimization and equitable distribution of closure errors, the strain ellipse in a fourth plane is determined for reasons which will appear clear in the next step. The fourth plane is chosen to make a tetrahedron with large dihedral angles. In Fig. 1, the acute spherical triangle formed by the intersection of the three data planes is shown. The pole to the fourth plane should be centrally located in this spherical triangle (if the specimen edges were spokes of an umbrella, the pole to the fourth plane would be its handle). The location of the pole to the fourth plane need not be determined with mathematical precision; all that is required is that the fourth plane should not intersect the spherical triangle of cropped great circles in Fig. 1.

The strain ellipse in the fourth plane is determined either graphically or mathematically using the three known stretches along its lines of intersection with the data planes. Because of the careful choice of a fourth plane, these stretches are never ambiguous even if the sectional ellipses are not quite compatible. Strain determination from three known stretches is a classical problem that has received much attention recently (De Paor 1986, 1988, Ragan 1987, Lisle & Ragan 1988). Ramsay (1967) used it to convert non-orthogonal sectional data into an orthogonal set, using a relatively cumbersome Mohr-circle construction. Simpler Mohr constructions are employed by Lisle & Ragan (1988). The most direct graphical solution is that of De Paor (1986) which employs the orthographic orientation net. However, because of the number of repetitions required here, the mathematical solution of Ragan (1987) or De Paor

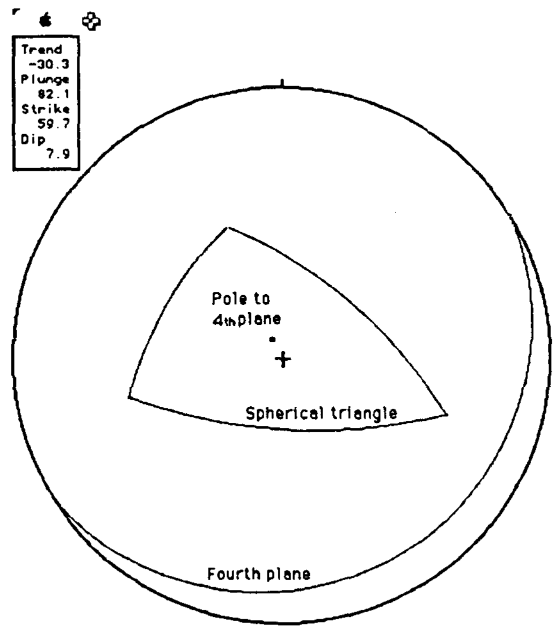

Fig. 1. Spherical triangle formed by three cropped great circles. The pole to the fourth plane is centrally located in the triangle so that the fourth plane's great circle does not pass close to any of the specimen edges. See text for explanation.

(1988) should be employed. The latter is summarized as follows. Let the three known reciprocal quadratic stretches in the fourth plane be $\lambda'_a, \lambda'_b, \lambda'_c$ in the directions of the intersection lines with the three data planes ϕ_a, ϕ_b, ϕ_c, measured as pitches in the fourth plane. Using the formulae derived by De Paor (1988), we have

$$\phi_1 = -\frac{1}{2}\arctan\left[\frac{\Sigma_{a,b,c}[(\lambda'_a - \lambda'_b)\cos 2\phi_c]}{\Sigma_{a,b,c}[(\lambda'_a - \lambda'_b)\sin 2\phi_c]}\right] \qquad (3)$$

$$\phi_2 = \phi_1 + 90°$$

$$\lambda'_i = \left[\frac{\lambda'_a \sec^2(\phi_a - \phi_j) - \lambda'_b \sec^2(\phi_b - \phi_j)}{\tan^2(\phi_a - \phi_j) - \tan^2(\phi_b - \phi_j)}\right], \qquad (4)$$

where the summation is over a permutation of the subscripts (replacing a by b, b by c, and c by a in turn) and $i = 1$ when $j = 2$ or vice versa. These two equations give the principal directions ϕ_1, ϕ_2 and their reciprocal quadratic stretches, λ'_1, λ'_2, in terms of known parameters of the intersection lines subscripted a, b and c. A computer listing to solve the equations is given in De Paor (1988).

It should be noted that equations (3) and (4) do not always give real solutions because every set of three radii does not lie on an ellipse; some lie on straight lines, parabola or hyperbola, for example (Fig. 2). Real data should generate ellipses but errors may occasionally create problems. While hyperbolae may be instantly identified as the outcome of flawed data, the real problem arises when bad data results in unreasonably large axial ratios. Also, if the three stretches are equal, equation (3) is undefined because a circle has no distinct principal directions. In this case, the radius of the circular section in the fourth plane is simply set equal to the three coincident stretch values.

Strain determination in test planes

The next step in the analysis is to inspect the strain state in a large number of planes perpendicular to the fourth plane (i.e. in planes containing the pole to the fourth plane) (Fig. 3), using the same strain-from-three-stretches algorithm that yielded the strain in the fourth plane (equations 3 and 4). Because the strain state is already known in four planes, not three, the solution is overdetermined, but only three of the four known strain ellipses are independent, so the four stretches in each test plane would lie on a single ellipse if the data sections were compatible. For each test section, three known stretches are selected for inclusion in equations (3) and (4). One is always the stretch in the fourth plane. The other two are taken from the test plane's intersections with two sides of the spherical triangle in Fig. 1. The test plane also intersects the third side of the spherical triangle externally (Fig. 3), but this data are not included in calculations. The reason for always choosing the stretch along the fourth plane and never the external intersection line is as follows. If the fourth plane had not been set up and an arbitrary set of test planes was chosen, then two problems would arise in attempting to determine test strain ellipses from the three stretches along the data plane intersections. First, the test plane and a data plane would intersect in a glancing angle whenever their two poles came close, with consequent loss of accuracy in estimation of the stretch along their intersection line. Second, the problem would become indeterminate whenever a test plane passed through a specimen edge, for then two incompatible stretches would be recorded for the same direction. Furthermore, accuracy would be poor in test planes close to the edge directions, both due to the small angle between two of the three known stretches and to incompatibility arising from proximity to the edge direction. Using the fourth plane, and given that the sides of the spherical triangle are never more than about 110° of arc nor less than about 70°, there are always large angles between the three known stretches and there are never glancing intersec-

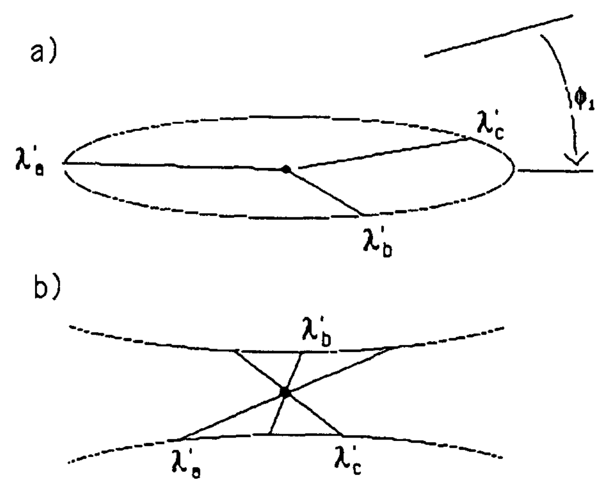

Fig. 2. Determination of strain from three known stretches. (a) Solution for good data. λ'_a, λ'_b and λ'_c are measured from the strike of the fourth plane (zero line). ϕ_1 is calculated as a positive clockwise angle from the zero line looking down on the plane. (b) Note that bad data may yield hyperbolae instead of an ellipse (or any other shape).

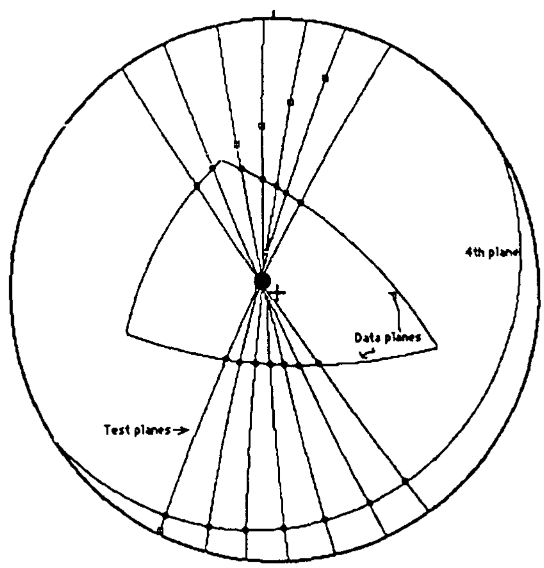

Fig. 3. Choice of a set of test planes intersecting in the pole to the fourth plane. Only a few planes are shown for clarity. Solid dots represent the three known stretches in each test plane. Squares are the external intersections referred to in the text.

tions with the chosen data planes. Whenever a test plane passes through a specimen edge direction, an average of the two incompatible stretches along the edge is employed; there are still three known stretches to work with, one average, one from a data plane and one from the fourth plane. A computer program called "Solve 3 stretches" (Appendix) speeds up the tedious exercise of determining sectional strain ellipses in a large number of test planes.

The sectional strain ellipses in the test planes thus generated do not lie on a perfect strain ellipsoid, rather they form three surface segments with jumps across the test planes containing the specimen edges (Fig. 4). Each of the three surface segments is a portion of a perfect ellipsoid and for small incompatibilities along the edge directions, the differences between the three ellipsoidal segments are not large. We therefore fit maximum and minimum axes to the near-ellipsoidal composite surface defined by the locus of test-section ellipses.

Location of the maximum and minimum stretch directions

Examination of Fig. 4 will convince the reader that the long axes of the sectional strain ellipses in test planes increase monotonically to a maximum in the test section that passes through the triaxial strain ellipsoid's longest axis (or closest to it in the case of relatively widely spaced test planes). Similarly, the short axis of the strain ellipsoid may be discovered by repeated testing of planes with ever-decreasing sectional short axes. Therefore, the test procedure may proceed in large increments (say 10°) at first; then when sectional axes stop getting longer and start shortening, or vice versa, the user may backtrack with a smaller interval between test sections (say 2°).

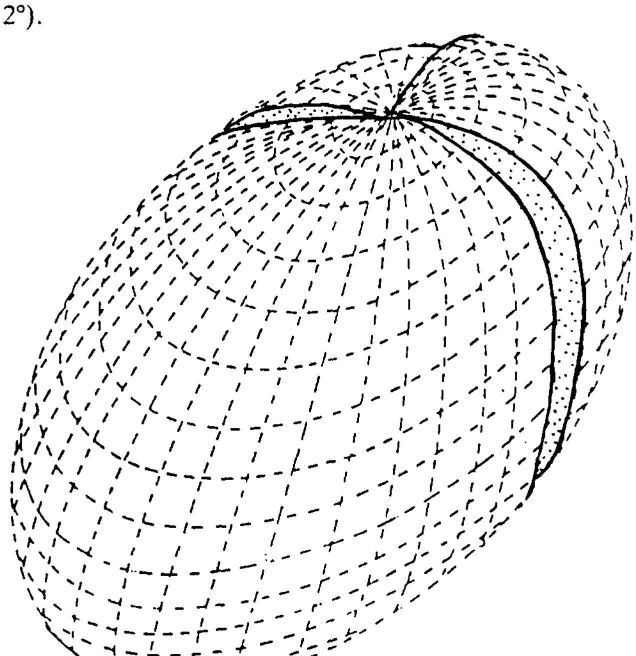

Fig. 4. A perspective view of a set of strain ellipses determined from test planes outlines the periphery of the three-dimensional strain ellipsoid. Shaded planes pass through the specimen edges and bound the three slightly mismatched divisions of the ellipsoid's surface.

Once the maximum and minimum triaxial stretches have been determined, the pole to their common plane is plotted and the test section passing through this pole is retrieved. The intermediate principal axis of the triaxial ellipsoid is determined using equation (2). Finally, the principal stretches S_1, S_2 and S_3 are all divided by $\sqrt[3]{\{S_1 * S_2 * S_3\}}$ in order to eliminate the artificial volumetric stretch introduced during the scaling procedure (step i). When the ellipsoid's axes have been determined and normalized in this way, they are plotted on a stereonet in order to check the angles between them. In theory, these angles must be 90° but in practice their value depends on the degree of compatibility of the data sections. It is important to note that the procedure described here identifies ellipsoid axes by their extreme stretch values while ignoring the requirement of orthogonality. It may be useful to plot loci of sectional principal directions in order to see whether choice of test planes to either side of the maximal and minimal cases might improve the orthogonality of axes without significantly affecting the axial ratio. Also useful for the evaluation of results are loci of equal stretch which instantly reveal the strain ellipsoid's symmetry class ($k > 1$ or $k < 1$).

MORE THAN THREE DATA PLANES

As pointed out by Owens (1984), confidence in a strain estimate can be increased by obtaining data from more than the minimum number of three section planes. Incorporation of extra data in the present technique is easily accommodated, in contrast to all previously published techniques except Owens's. The extra planes provide a fifth, or subsequent, known stretch direction in each test plane. Since equations (3) and (4) are then overdetermined, the sectional strain ellipse is most conveniently determined graphically using the technique of De Paor (1986). Graphical plotting of the data for each plane has the advantage that it reveals inaccuracies due to incorrect data. As before, stretch estimates from glancing intersections between the test plane and extra data planes must be treated with scepticism.

INVERSE PROCEDURE

It is sometimes necessary to carry out the inverse to the above procedure, that is to determine the two-dimensional strain state in a plane given the three-dimensional strain ellipsoid (Ramberg 1976, Gendzwill & Stauffer 1981). For example, one may wish to determine the map-plane strain ellipse when the axes of the strain ellipsoid are all oblique to the horizontal. Using the method presented here, this is not a separate problem but rather a part of the same procedure. The simplest solution is to treat the known principal planes of the strain ellipsoid as three 'data' planes and to determine the strain in any other 'test' plane using equations (3) and (4). If the test plane makes a glancing angle with any of the principal planes or if it passes through or close to a principal direction, then it is necessary to first determine the strain state in a fourth plane that intersects the test plane more suitably.

The inverse procedure provides a way to test the accuracy of a solution, by determining from the calculated principal stretches, the axial ratios and long axis orientations for the data planes, for comparison with the observed data.

CONCLUSIONS

The determination of three-dimensional strain states from two-dimensional data need not present an insurmountable barrier. Using recently published equations for strain from three known stretches, the strain state in an arbitrary test plane may be determined from the three known stretches along its lines of intersection with three data planes on which two dimensional strain analyses have been performed. By examining a set of coaxial test planes, the triaxial strain state may be revealed. Procedures outlined above ensure that errors due to glancing angles of intersection, near-parallel data planes or congruence of two stretches are avoided. Extra data planes may be included to increase confidence in the calculated test sectional ellipses and the procedure may be reversed in order to evaluate its accuracy.

The programs referred to in the text are available to readers who send a blank Macintosh diskette and a stamped self-addressed mailing package to the author.

Acknowledgements—Thanks to Bill Owens, Richard Lisle and Sue Treagus for reviews.

REFERENCES

De Paor, D. G. 1986. Orthographic analysis of geological structures—II. Practical applications. *J. Struct. Geol.* **8**, 87–100.

De Paor, D. G. 1988. Strain from three known stretches—an exact solution. *J. Struct. Geol.* **10**, 639–642.

Gendzwill, D. J. & Stauffer, M. R. 1981. Analysis of triaxial ellipsoids: their shapes, plane sections and plane projections. *Maths. Geol.* **13**, 135–152.

Helm, D. G. & Siddans, A. W. B. 1971. Deformation of a slaty, lapillar tuff in the English Lake District: discussion. *Bull. geol. Soc. Am.* **82**, 523–531.

Lisle, R. J. 1985. *Geological Strain Analysis. A Manual for the R_f/ϕ Technique*. Pergamon Press, London.

Lisle, R. J. & Ragan, D. M. 1988. Strain from three stretches—a simple Mohr circle solution. *J. Struct. Geol.* **10**, 905–906.

Milton, N. J. 1980. Determination of the strain ellipsoid from measurements on any three sections. *Tectonophysics* **64**, T19–T27.

Oertel, G. 1978. Strain determination from the measurement of pebble shapes. *Tectonophysics* **50**, T1–T7.

Owens, W. H. 1984. The calculation of a best-fit ellipsoid from elliptical sections on arbitrarily oriented planes. *J. Struct. Geol.* **6**, 571–578.

Ragan, D. M. 1987. Strain from three measured stretches. *J. Struct. Geol.* **9**, 897–898.

Ramberg, H. 1976. The strain in a sheet intersecting the strain ellipsoid at any angle. *Bull. Soc. géol. Fr.* **18**, 1417–1422.

Ramsay, J. G. 1967. *Folding and Fracturing of Rocks*. McGraw-Hill, New York.

Sanderson, D. J. 1976. The superposition of compaction and plane strain. *Tectonophysics* **30**, 35–54.

Shimamoto, T. & Ikeda, Y. 1976. A simple algebraic method for strain estimation from deformed ellipsoidal objects—I. Basic theory. *Tectonophysics* **36**, 315–337.

Treagus, S. H. 1986. Some applications of the Mohr diagram for three dimensional strain. *J. Struct. Geol.* **8**, 819–830.

APPENDIX

DETAILS OF COMPUTER PROGRAMS

The programs available from the author run on any Macintosh microcomputer and employ the standard user interface, therefore they are easily used even by non-computer oriented persons, i.e. most geologists.

Figure A1 illustrates the screen layout for "2D–3D Compatibility" which is divided into four panels, one for each section plane and one for the stereonet. The strike and dip of each plane is entered into editable fields along with the pitch of each sectional ellipse axis. The user may then choose to enter absolute values of the sectional ellipse axes S_1 and S_2, in which case the computer calculates the axial ratios $R_s = S_1/S_2$ and area stretch factors $A_s = S_1 * S_2$ and enters them in the sixth and seventh edit fields of each panel. Alternatively, the user may enter the strain ratios, and let the computer determine the absolute stretches from equations (1). Since the area stretch factors are given a default value of 1, this generally results in the display of three incompatible sectional ellipses. When the "Test compatibility" menu option is selected, the section plane orientations are illustrated on the stereonet along with points indicating the axial pitches. In each panel, an ellipse is drawn to represent the sectional strain state, and two lines through its center represent the stretches determined for the specimen edges (directions of plane intersections). On a Mac II, these lines, ellipses, great circles and the numbers in edit fields are color-coded to assist in identification of features of each section plane, but even in black and white, it is immediately clear if one ellipse as scaled is too large or too small to fit the other two. Then the area stretch factor $S_1 * S_2$ is changed in the appropriate plane and the test of compatibility is re-selected. If an ellipse displays errors of opposite sign (i.e. if it is too large to fit one section but too small to fit the other), no amount of rescaling will eliminate the error, but one should divide the incompatibility evenly between the two edge directions, assuming an equal level of confidence in the strain data from the three section planes. These errors can be closed by adjusting the sectional principal directions, but such modification of the data is potentially dangerous and should always be limited to a few degrees. Note that closure of errors of like sign on both edges by adjustment of calculated axial ratios is never warranted since area stretch factors can always achieve the same end result.

Once a satisfactory fit of the three sectional ellipses has been achieved, the program is quit and the edit field data is automatically stored to a text file for subsequent use. Prior to quitting, the screen may be saved to a MacPaint file for printing.

In order to help the user to visualize the procedure, a second program, "2D–3D Perspective" is presented in Fig. A2. This program displays the sectional strain ellipses in perspective view in their respective great circles, using orthographic rather than stereographic hemispheric projection. When the three section planes have been entered, the great circles are cropped at their lines of intersection to reveal incompatibilities in the strain ellipse radii along the specimen edges. The area stretch factors obtained from the first program may be applied to the cropped perspective views of the ellipses.

A third program, "Solve 3 stretches", is used to solve equations (3) and (4) repeatedly for the test planes. It requires as input, three stretches from the test plane's intersections with the fourth plane and two of the data planes as explained in the main text. Its output gives the sectional stretches and principal directions.

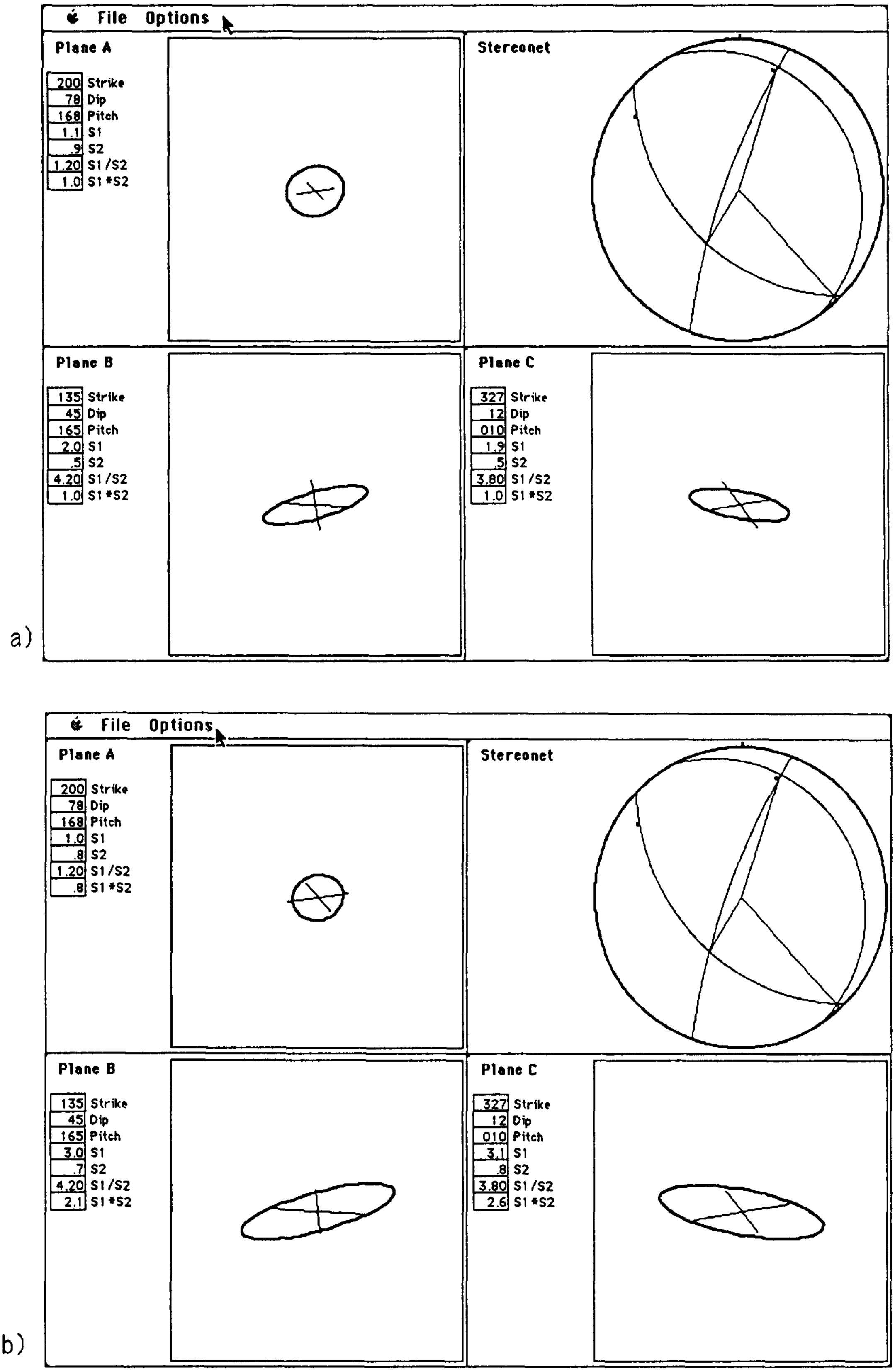

Fig. A1. Screen layout for program "2D–3D Compatibility": (a) after entering data and testing its compatibility; (b) after applying area stretch factors ($S_1 * S_2$) to maximize compatibility.

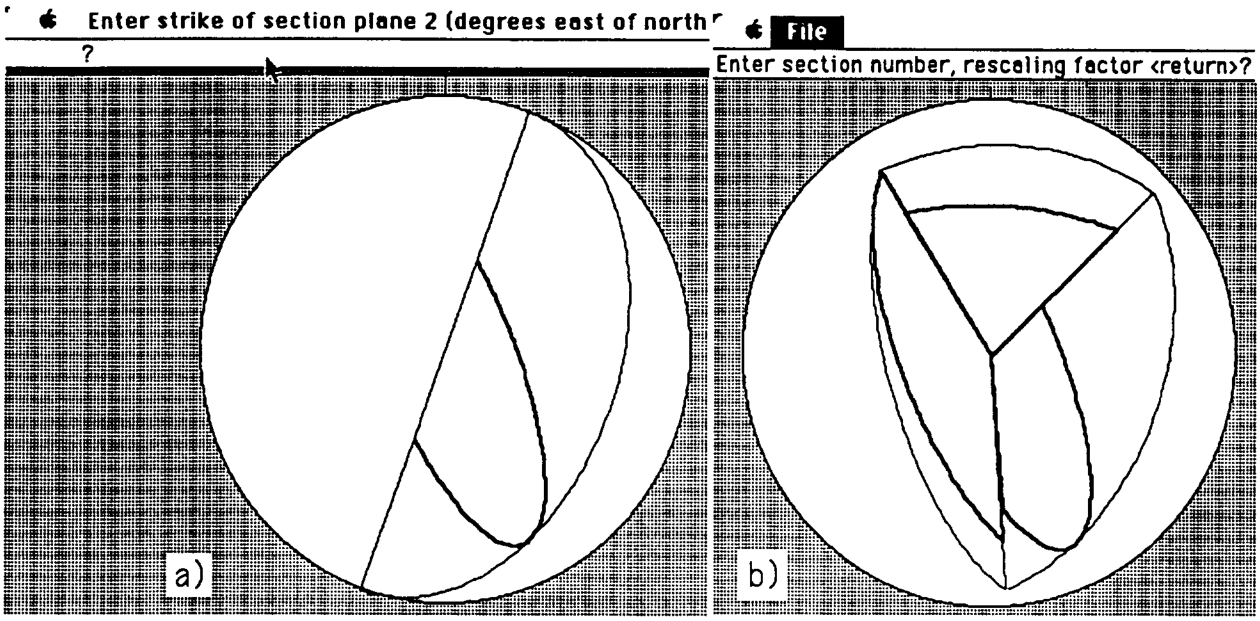

Fig. A2. Output of program "2D–3D Perspective". (a) Data entered for one section plane. The strain ellipse is shown in perspective view in this plane. (b) Cropped view of the three section planes and their lines of intersection. The ratios used are highly incompatible, for effect.